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Five-dimensional supersymmetric Yang-Mills theories and random plane partitions

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Abstract: Five-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theories are investigated from the viewpoint of random plane partitions. It is shown that random plane partitions are factorizable as q-deformed random partitions so that they admit the interpretations as five-dimensional Yang-Mills and as topological string amplitudes. In particular, they lead to the exact partition functions of five-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills with the Chern-Simons terms. We further show that some specific partitions, which we call the ground partitions, describe the perturbative regime of the gauge theories. We also argue their role in string theory. The gauge instantons give the deformation of the ground partition.

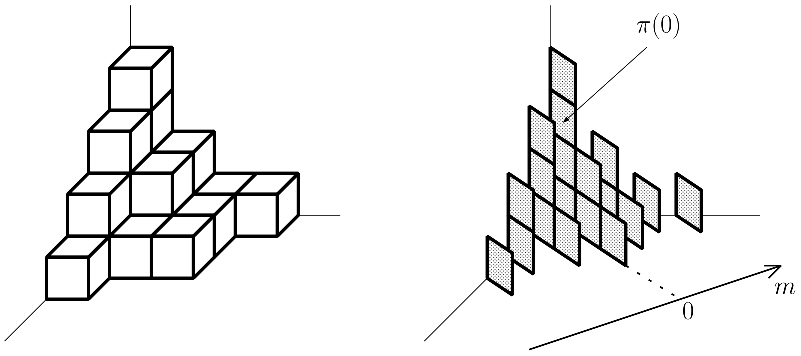
Keywords: Field Theories in Higher Dimensions, Nonperturbative Effects, Superstring Vacua, Statistical Methods.

Contents

1. Introduction and summary

Recently it becomes possible [1, 2] to compute the exact partition functions of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. Remarkably, the celebrated Seiberg-Witten solutions [3] of the gauge theories emerge [2, 4] through the statistical models of random partitions.

$\mathcal{N} = 2$ supersymmetric gauge theories are realized [5] by type-IIA strings on certain Calabi-Yau threefolds. The topological vertex [6] provides a powerful method to compute all genus topological A-model partition functions on local toric Calabi-Yau threefolds. The string amplitudes on the relevant threefolds are computed [7, 8] by using this and shown to reproduce the gauge instanton contributions to the exact partition functions of the gauge theories. It turns out surprisingly that the topological vertex is identified [9] with the partition function of a model of random plane partitions [10]. This interpretation is further explored in [11]. The appearance of random plane partitions in string theory suggests their possible relation to the gauge theories.



(a) (b)

Figure 1: The three-dimensional Young diagram (a) and the corresponding sequence of partitions (the two-dimensional Young diagrams) (b).

Nevertheless, the exact solutions of the gauge theories have not been studied from the viewpoint of random plane partitions. In this article, we investigate five-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theories from the perspective of random plane partitions. A plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is an array of non-negative integers satisfying $\pi_{ij} \geq \pi_{i+1j}$ and $\pi_{ij} \geq \pi_{ij+1}$, and identified with the three-dimensional Young diagram as depicted in figure 1a. The diagram is also regarded as a sequence of partitions $\pi(m)$, where $m \in \mathbb{Z}$. See figure 1b. Among the series of partitions, $\pi(0)$ will be called the main diagonal partition of π. We consider the following model of random plane partitions.

$$Z(q,Q) = \sum_{\pi} q^{|\pi|}\, Q^{|\pi(0)|}\,, \tag{1.1}$$

where q and Q are indeterminates. $|\pi|$ and $|\pi(0)|$ denote respectively the total numbers of cubes and boxes of the corresponding diagrams. By an identification of q and Q with the relevant string theory parameters, the partition function can be converted into topological A-model string amplitude on the local Calabi-Yau threefold $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbb{P}^1$.

The above model has an interpretation as random partitions. It can be seen by rewriting the partition function as

$$Z(q,Q) = \sum_{\mu} Q^{|\mu|} \left(\sum_{\pi(0)=\mu} q^{|\pi|} \right), \tag{1.2}$$

where the summation over plane partitions in (1.1) are divided into two branches. The partitions μ are thought as the ensemble of the model by summing first over the plane partitions whose main diagonal partitions are μ. As the model of random partitions, it becomes a q-deformation of the standard Plancherel measured random partitions.

It is well known that partitions are realized as states of two-dimensional free fermions. By using the folding [12] of a single complex fermion into N-component ones, each partition can be expressed as a set of N charged partitions $(\lambda^{(r)}, p_r)$, where $\lambda^{(r)}$ are partitions and p_r are the U(1) charges of the N-component fermions. This correspondence is used [2] to study the exact partition functions for four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories. Among partitions, these coming from the charged empty partitions $(\lambda^{(r)}, p_r) = (\emptyset, p_r)$ turn out to play special roles. We call them the ground partitions. Regarding the model as the q-deformed random partitions, we write the Boltzmann weights for the ground partitions by $Z_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\}, Q)$. We will factor the partition function into

$$Z(q,Q) = \sum_{\{p_r\}} Z_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\}, Q) \sum_{\{\lambda^{(r)}\}} Z_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\}, \{p_r\}, Q)\,. \tag{1.3}$$

This factorization turns out useful to find out the gauge theoretical interpretations. The relevant field theory parameters are a_r, Λ and R, where a_r are the vacuum expectation values of the adjoint scalar in the vector multiplet and Λ is the scale parameter of the underlying four-dimensional theory. R is the radius of S^1 in the fifth dimension. We identify these parameters with q, Q and p_r in (1.3) as follows;

$$q = e^{-\frac{2}{N}R\hbar}\,, \qquad Q = -(2R\Lambda)^2\,, \qquad \widetilde{p}_r = \frac{a_r}{\hbar}\,, \tag{1.4}$$

where $\widetilde{p}_r = p_r + \frac{1}{N}(r - \frac{N+1}{2})$. The parameter $\hbar$, which can be thought as a chemical potential for plane partitions, is often identified with the string coupling constant g_{st}. Via the above identifications we derive

$$Z_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\}, Q) \sum_{\{\lambda^{(r)}\}} Z_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\}, \{p_r\}, Q) = Z_{5d\,\mathrm{SYM}}^{\mathrm{SU}(N)}(\{a_r\};\, \Lambda, R, \hbar)\,, \tag{1.5}$$

where the r.h.s. is the exact partition function [2] for five-dimensional $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N)$ Yang-Mills with the Chern-Simons term [13]. The five-dimensional theory is living on $\mathbb{R}^4 \times S^1$. The Chern-Simons coupling constant c_{cs} is quantized to N. The above can be said more precisely as follows;

$$Z_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\},\, Q) \;=\; Z_{5d\,\mathrm{SYM}}^{\mathrm{SU}(N)\,\mathrm{pert}}(\{a_r\};\, \Lambda, R, \hbar)\,, \tag{1.6}$$

$$\sum_{\{\lambda^{(r)}\}} Z_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\}, \{p_r\},\, Q) \;=\; Z_{5d\,\mathrm{SYM}}^{\mathrm{SU}(N)\,\mathrm{inst}}(\{a_r\};\, \Lambda, R, \hbar)\,, \tag{1.7}$$

where the r.h.s. s are respectively the perturbative part and the instanton part of the exact partition function.

The gauge theory partition function is obtained as the $\hbar \to 0$ limit of the above partition function. The ground partition in (1.5) becomes very large since its size is $\sim p_r^2$ and the U(1) charges p_r scale as $\hbar^{-1}$ at the field theory limit. The equality (1.6) shows that these ground partitions describe the perturbative regime of the Coloumb branch. $\lambda^{(r)}$ in (1.5) represent U(1) instantons with $c_2 = |\lambda^{(r)}|$ on a non-commutative $\mathbb{R}^4$ [14]. They can be viewed as excitations from the ground partition. These excitations emerge as the quantum (non-perturbative) deformation of the perturbative Coloumb branch when $\lambda^{(r)}$ are comparable to the ground partition.

We start section 2 with a brief review on the transfer matrix approach [10] to random plane partitions. The perspective of the q-deformation is emphasized. In section 3 we examine the factorization (1.3) for the case of SU(2) and confirm the equality (1.7). Section 4 is devoted to prove the equality (1.6) for this case. The discussion is also applicable to the cases of the higher rank gauge groups. In section 5 we treat the cases of $\mathrm{SU}(N)$. The instanton parts can be converted to topological A-model string amplitudes on the relevant local Calabi-Yau threefolds. These amplitudes are summarized in appendix.

2. Random plane partitions and $\mathrm{U}(1)$ gauge theory

A plane partition π is an array of non-negative integers

$$\begin{array}{cccc} \pi_{11} & \pi_{12} & \pi_{13} & \cdots \\ \pi_{21} & \pi_{22} & \pi_{23} & \cdots \\ \pi_{31} & \pi_{32} & \pi_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{array} \tag{2.1}$$

satisfying $\pi_{ij} \geq \pi_{i+1j}$ and $\pi_{ij} \geq \pi_{ij+1}$ for all $i, j \geq 1$. Plane partitions are identified with the three-dimensional Young diagrams. The three-dimensional diagram π is a set of unit cubes such that π_{ij} cubes are stacked vertically on each (i, j)-element of π. The size of π is $|\pi| \equiv \sum_{i,j\geq 1} \pi_{ij}$, which is the total number of cubes of the diagram. Each diagonal slice of π becomes a partition, that is, a sequence of weakly decreasing non-negative integers. Let $\pi(m)$ be a partition along the m-th diagonal slice.

$$\pi(m) = \begin{cases} (\pi_{1\; m+1}, \pi_{2\; m+2}, \pi_{3\; m+3}, \cdots) & \text{for } m \geq 0 \\ (\pi_{-m+1\; 1}, \pi_{-m+2\; 2}, \pi_{-m+3\; 3}, \cdots) & \text{for} m \leq -1\,. \end{cases} \tag{2.2}$$

In particular $\pi(0) = (\pi_{11} \geq \pi_{22} \geq \pi_{33} \geq \cdots \geq 0)$ is the main diagonal partition. The series of partitions $\pi(m)$ satisfies the condition

$$\cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots\,, \tag{2.3}$$

where $\mu \succ \nu$ means the following interlace relation between two partitions $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0)$ and $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq 0)$

$$\mu \succ \nu \quad \Longleftrightarrow \quad \mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \mu_3 \geq \cdots\,. \tag{2.4}$$

We have $|\pi| = \sum_{m=-\infty}^{+\infty} |\pi(m)|$, where the size of a partition μ is also denoted by $|\mu| \equiv \sum_{i\geq 1} \mu_i$.

2.1 Random plane partitions

A model of random plane partitions relevant to describe five-dimensional $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(1)$ gauge theory is defined by the following partition function.

$$Z_q^{\mathrm{U}(1)}(Q) \equiv \sum_{\pi} q^{|\pi|} Q^{|\pi(0)|}\,. \tag{2.5}$$

The Boltzmann weight consists of two parts. The first contribution comes from the energy of plane partitions π, and the second contribution is a chemical potential for the main

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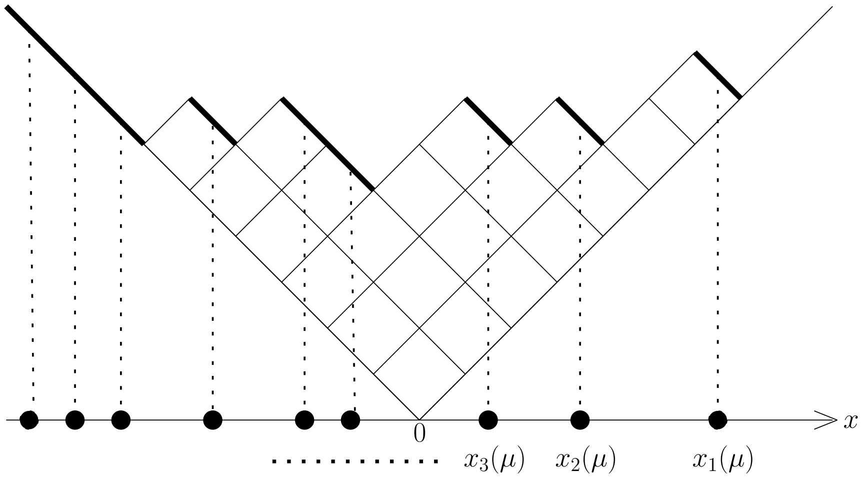

Figure 2: The correspondence between the Maya diagram and the Young diagram of $\mu = (7, 5, 4, 2, 2, 1)$. Elements of the Maya diagram are denoted by $\bullet$.

diagonal partitions $\pi(0)$. To contact with the U(1) gauge theory we will identify the indeterminates q and Q with the following field theory parameters.

$$q = e^{-2R\hbar}, \qquad Q = -(2R\Lambda)^2, \tag{2.6}$$

where R is the radius of S^1 in the fifth dimension and Λ denotes the scale parameter of the underlying four-dimensional theory. The parameter $\hbar$ is often identified with string coupling constant g_{st}.

2.1.1 Transfer matrix approach

The transfer matrix approach [10] allows us to express the random plane partitions (2.5) in terms of two-dimensional conformal field theory ($2d$ free fermion system). The partition function can be computed exactly by using the standard technique of $2d$ CFT.

It is well known that partitions are realized as states of $2d$ free fermions by using the Maya diagrams. Let $\psi(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k z^{-k-\frac{1}{2}}$ and $\psi^*(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k^* z^{-k-\frac{1}{2}}$ be complex fermions with the anti-commutation relations

$$\{\psi_k, \psi_l^*\} = \delta_{k+l,0}\,, \qquad \{\psi_k, \psi_l\} = \{\psi_k^*, \psi_l^*\} = 0\,. \tag{2.7}$$

Let $\mu = (\mu_1, \mu_2, \ldots)$ be a partition. The Maya diagram is the set S_μ which is defined by

$$S_\mu \equiv \left\{ x_i(\mu) = \mu_i - i + \frac{1}{2}\,;\ i \geq 1 \right\} \qquad \left(\subset \mathbb{Z} + \frac{1}{2} \right). \tag{2.8}$$

The correspondence with the Young diagram is depicted in figure 2. By using the Maya diagram the partition can be mapped to the following fermion state.

$$|\mu; n\rangle = \psi_{-x_1(\mu)-n}\psi_{-x_2(\mu)-n}\cdots\psi_{-x_{l(\mu)}(\mu)-n}\psi^*_{-l(\mu)+\frac{1}{2}+n}\psi^*_{-l(\mu)+\frac{3}{2}+n}\cdots\psi^*_{-\frac{1}{2}+n}|\emptyset; n\rangle\,, \tag{2.9}$$

where $l(\mu)$ is the length of μ, that is, the number of the non-zero μ_i. In (2.9) the state $|\emptyset; n\rangle$ is the ground state of the charge n sector. It is defined by the conditions

$$\begin{aligned} \psi_k|\emptyset; n\rangle &= 0 \qquad \text{for } k > -n\,, \\ \psi_k^*|\emptyset; n\rangle &= 0 \qquad \text{for } k > n\,. \end{aligned} \tag{2.10}$$

We mainly consider the $n = 0$ sector in the below.

The basic ingredient of the transfer matrix approach is the following evolution operator at a discretized time $m \in \mathbb{Z}$.

$$\Gamma(m) \equiv \begin{cases} \exp\left(\sum_{k=1}^{+\infty} \frac{1}{k} q^{k(m+\frac{1}{2})} J_{-k}\right) & \text{for } m \geq 0 \\ \exp\left(\sum_{k=1}^{+\infty} \frac{1}{k} q^{-k(m+\frac{1}{2})} J_{k}\right) & \text{for } m \leq -1\,, \end{cases} \tag{2.11}$$

where $J_{\pm k}$ are the modes of the standard U(1) current

$$:\psi\psi^*:(z) = \sum_n J_n z^{-n-1}\,. \tag{2.12}$$

Implications of the above operators in random plane partitions can be understood from their matrix elements: For $m \geq 0$,

$$\langle \mu; 0|\Gamma(m)|\nu; 0\rangle = \begin{cases} q^{(m+\frac{1}{2})(|\mu|-|\nu|)} & \mu \succ \nu \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.13}$$

and for $m \leq -1$,

$$\langle \mu; 0|\Gamma(m)|\nu; 0\rangle = \begin{cases} q^{(m+\frac{1}{2})(|\mu|-|\nu|)} & \mu \prec \nu \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.14}$$

It follows from (2.13) and (2.14) that the partition function is expressed as

$$Z_q^{\mathrm{U}(1)}(Q) = \langle \emptyset; 0| \left\{ \prod_{m \leq -1} \Gamma(m) \right\} Q^{L_0} \left\{ \prod_{m \geq 0} \Gamma(m) \right\} |\emptyset; 0\rangle\,. \tag{2.15}$$

This expression allows us to compute the partition function by using the standard technique of $2d$ CFT. It becomes

$$Z_q^{\mathrm{U}(1)}(Q) = \prod_{n=1}^{+\infty} \frac{1}{(1 - Qq^n)^n}\,. \tag{2.16}$$

It is also possible to regard $\hbar$ as a chemical potential for plane partitions. Thermodynamic limit of (2.5) is obtained by letting $\hbar \to 0$. Let us recall that the mean values of $|\pi|$ and $|\pi(0)|$ are respectively given by $q\frac{\partial}{\partial q} \ln Z_q^{\mathrm{U}(1)}$ and $Q\frac{\partial}{\partial Q} \ln Z_q^{\mathrm{U}(1)}$. It follows from (2.16) that they behave $\langle|\pi|\rangle = O(\hbar^{-3})$ and $\langle|\pi(0)|\rangle = O(\hbar^{-2})$ as $\hbar \to 0$ ($q \to 1$). Therefore a typical plane partition π at the limit $\hbar \to 0$ is a plane partition of order $\hbar^{-3}$, and its main diagonal partition $\pi(0)$ becomes a partition of order $\hbar^{-2}$.

2.1.2 q-deformed random partitions

We can interpret the random plane partitions (2.5) as a model of random partitions. It is identified with a q-deformation of the standard Plancherel measured random partitions. (The deformation is different from [2].) To see this, we rewrite the partition function by using the Schur functions. An insertion of the unity $1 = \sum_\mu |\mu;0\rangle\langle\mu;0|$ factorizes (2.15) into

$$Z_q^{\mathrm{U}(1)}(Q) = \sum_\mu Q^{|\mu|} \langle\emptyset;0| \prod_{m\le -1} \Gamma(m)|\mu;0\rangle\langle\mu;0| \prod_{m\ge 0}\Gamma(m)|\emptyset;0\rangle\,. \tag{2.17}$$

The matrix elements in the above turn to be

$$\begin{aligned}\langle\emptyset;0|\prod_{m\le -1}\Gamma(m)|\mu;0\rangle &= \langle\emptyset;0|\prod_{k=1}^{+\infty}\exp\left(\frac{1}{k}\sum_{i=1}^{+\infty} q^{k(i-\frac{1}{2})}J_k\right)|\mu;0\rangle \\ &= s_\mu(q^{\frac{1}{2}}, q^{\frac{3}{2}},\ldots)\,, \end{aligned}\tag{2.18}$$

$$\begin{aligned}\langle\mu;0|\prod_{m\ge 0}\Gamma(m)|\emptyset;0\rangle &= \langle\mu;0|\prod_{k=1}^{+\infty}\exp\left(\frac{1}{k}\sum_{i=1}^{+\infty} q^{k(i-\frac{1}{2})}J_{-k}\right)|\emptyset;0\rangle \\ &= s_\mu(q^{\frac{1}{2}}, q^{\frac{3}{2}},\ldots), \end{aligned}\tag{2.19}$$

where $s_\mu(q^{\frac{1}{2}}, q^{\frac{3}{2}},\ldots)$ is the Schur function $s_\mu(x_1,x_2,\cdots)$ specialized at $x_i = q^{i-\frac{1}{2}}$ $(i\ge 1)$. Therefore we obtain

$$Z_q^{\mathrm{U}(1)}(Q) = \sum_\mu Q^{|\mu|} s_\mu(q^{-\rho})^2\,, \tag{2.20}$$

where the multiple index $\rho \equiv (-\frac{1}{2}, -\frac{3}{2},\cdots,-i+\frac{1}{2},\cdots)$ is used. The expression (2.20) allows us to interpret (2.5) as a model of q-deformed random partitions. It is also clear from (2.17) that partitions μ are the main diagonal partitions $\pi(0)$.

The four-dimensional limit of the model is obtained by letting $R \to 0$ under the identification (2.6). To obtain the limit of (2.20) the following product formula of the Schur function [15] becomes useful.

$$s_\mu(q^{-\rho}) = q^{-\frac{1}{4}\kappa(\mu)} \prod_{(i,j)\in\mu} \frac{1}{q^{-\frac{h(i,j)}{2}} - q^{\frac{h(i,j)}{2}}}\,, \tag{2.21}$$

where $h(i,j)$ is the hook length of the box (i,j) in the Young diagram, and $\kappa(\mu) \equiv 2\sum_{(i,j)\in\mu}(j-i)$. By using (2.21) we can see from (2.20) that

$$\lim_{R\to 0} Z_q^{\mathrm{U}(1)}(Q) = \sum_\mu \left(\frac{\Lambda}{\hbar}\right)^{2|\mu|} \frac{(-1)^{|\mu|}}{(\prod_{(i,j)\in\mu} h(i,j))^2}\,. \tag{2.22}$$

This is the random partitions [2, 4] which describes four-dimensional $\mathcal{N}=2$ supersymmetric U(1) gauge theory on non-commutative $\mathbb{R}^4$.

2.1.3 Interpretation as topological string amplitude

The partition function (2.5) can be converted to topological string amplitude on a certain non-compact Calabi-Yau threefold by identifying q and Q with the string theory parameters. Let us consider the topological A-model on $\mathcal{O}\oplus\mathcal{O}(-2)\to\mathbb{P}^1$. All genus A-model

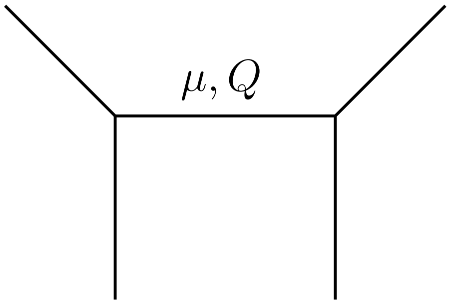

Figure 3: The diagram for $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbb{P}^1$. $Q = e^{-t}$, where t is the Kähler parameter of the $\mathbb{P}^1$. Partitions μ are attached to the $\mathbb{P}^1$.

partition function on this local geometry is computed [16] from the M-theory viewpoint. It is given by

$$Z^{\mathrm{U}(1)}_{\mathrm{string}}(q, Q) = \prod_{n=1}^{\infty} \frac{1}{(1 - Qq^n)^n}\,, \tag{2.23}$$

where the string coupling constant g_{st} and the Kähler parameter t of the $\mathbb{P}^1$ appear as $q = e^{-g_{st}}$ and $Q = e^{-t}$.

The topological vertex [6] makes it possible to compute all genus A-model partition functions on local toric Calabi-Yau threefolds by using diagrammatic techniques like the Feynman rules, where the diagrams are the dual toric diagrams of the local geometries. The diagram for the above local geometry is described in figure 3. A computation using the topological vertices presents the amplitude in the same form as (2.20). In that expression, partitions μ are attached to the $\mathbb{P}^1$.

3. SU(2) Yang-Mills and random plane partitions

3.1 Multi-component fermion

The Fock representation of a single complex fermion has [12] an alternative realization by exploiting N complex fermions:

$$\psi^{(r)}_k,\ \psi^{(s)*}_l,\ k, l \in \mathbb{Z} + \frac{1}{2},\ r, s = 1, 2, \ldots, N\,, \tag{3.1}$$

with the anti-commutation relations

$$\{\psi^{(r)}_k, \psi^{(s)*}_l\} = \delta_{r,s}\delta_{k+l,0}\,. \tag{3.2}$$

$\psi^{(r)}(z)$ and $\psi^{(s)*}(z)$ are obtained from $\psi(z)$ and $\psi^*(z)$ by the following identifications.

$$\psi^{(r)}_k = \psi_{N(k-\xi_r)}\,, \qquad \psi^{(s)*}_l = \psi_{N(l+\xi_s)}\,, \tag{3.3}$$

where $\xi_r \equiv \frac{1}{N}(r - \frac{N+1}{2})$.

It is convenient to consider partitions paired with the U(1) charges. We denote such a charged partition by (μ, n), where μ is a partition and n is the U(1) charge. The states $|\mu;\, n\rangle$ constitute bases of the Fock space of a single complex fermion. Thanks to the above realization of N fermions, we can express (μ, n) uniquely by means of N charged partitions $(\lambda^{(r)}, p_r)$ and vice versa through the identification

$$|\mu;\, n\rangle = \bigotimes_{r=1}^{N} |\lambda^{(r)};\, p_r\rangle_{(r)}\,. \tag{3.4}$$

The conservation of the U(1) charges implies $\sum_{r=1}^{N} p_r = n$. The correspondence between the partitions μ and $\lambda^{(r)}$ can be described explicitly by using the Maya diagrams (shifted by the U(1) charges). It becomes

$$\Big\{x_i(\mu) + n\,;\ i \geq 1\Big\} = \bigcup_{r=1}^{N} \Big\{N(x_{i_r}(\lambda^{(r)}) + \widetilde{p}_r)\,;\ i_r \geq 1\Big\}\,, \tag{3.5}$$

where $\widetilde{p}_r \equiv p_r + \xi_r$. In particular, when $n = 0$, the following information on the partitions is obtainable from the above correspondence by applying the method of power-sums [2].

$$\begin{aligned}
|\mu| &= N\sum_{r=1}^{N}|\lambda^{(r)}| + \frac{N}{2}\sum_{r=1}^{N} p_r^2 + \sum_{r=1}^{N} r p_r\,,\\
\kappa(\mu) &= N^2\sum_{r=1}^{N}\kappa(\lambda^{(r)}) + 2N^2\sum_{r=1}^{N}\widetilde{p}_r|\lambda^{(r)}| + \frac{N^2}{3}\sum_{r=1}^{N}\widetilde{p}_r^{\,3}\,.
\end{aligned} \tag{3.6}$$

The Maya diagram $S_{\lambda^{(r)}}$ is considered as the subset of S_μ by (3.5). It is also a subset of $N\mathbb{Z} + r - \frac{1}{2}$ $(\subset \mathbb{Z} + \frac{1}{2})$. We may colour the set $N\mathbb{Z} + r - \frac{1}{2}$ just by attaching the number r to all the elements. $S_{\lambda^{(r)}}$ gets coloured by r. (3.5) shows that S_μ has N colours, where the number r is attached to these elements coming from $S_{\lambda^{(r)}}$.

3.2 SU(2) Yang-Mills from random plane partitions

Let us begin with the case of two-component fermions. Owing to the identification (3.4) any charged partition (μ, n) can be expressed by $(\lambda^{(1)}, p_1)$ and $(\lambda^{(2)}, p_2)$. We consider the neutral case, that is, $n = 0$. $(\lambda^{(1)}, p)$ and $(\lambda^{(2)}, -p)$ determine a partition in the neutral sector. We denote such a partition by $\mu(\lambda^{(1)}, \lambda^{(2)};\, p)$. The corresponding Maya diagram $S_{\mu(\lambda^{(1)},\lambda^{(2)};p)}$ can be read from (3.5) as

$$S_{\mu(\lambda^{(1)},\lambda^{(2)};p)} = \left\{2\left(x_{i_1}(\lambda^{(1)}) + p - \frac{1}{4}\right);\ i_1 \geq 1\right\} \cup \left\{2\left(x_{i_2}(\lambda^{(2)}) - p + \frac{1}{4}\right);\ i_2 \geq 1\right\}. \tag{3.7}$$

In the present case, the quantities (3.6) turn to be

$$\begin{aligned}
|\mu(\lambda^{(1)}, \lambda^{(2)}; p)| &= 2(|\lambda^{(1)}| + |\lambda^{(2)}|) + p(2p-1)\,,\\
\kappa(\mu(\lambda^{(1)}, \lambda^{(2)}; p)) &= 4(\kappa(\lambda^{(1)}) + \kappa(\lambda^{(2)})) + 2(4p-1)(|\lambda^{(1)}| - |\lambda^{(2)}|)\,.
\end{aligned} \tag{3.8}$$

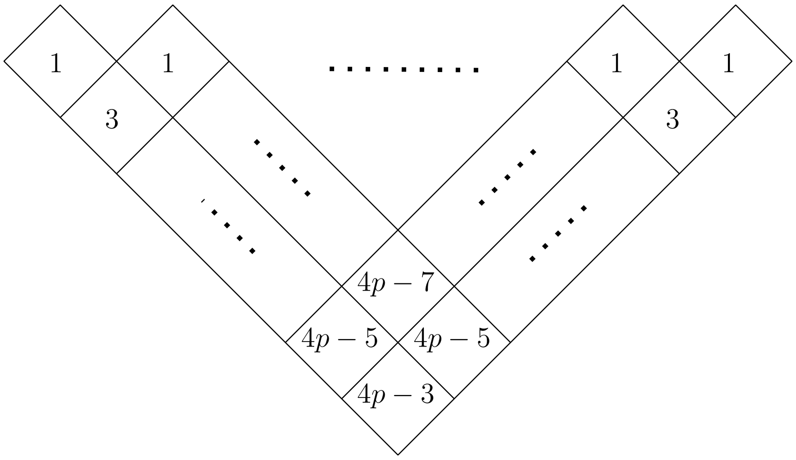

Figure 4: The SU(2) ground partition for $p \geq 1$. It is the partition $(2p-1, 2p-2, \cdots, 1)$. The numbers in the boxes are the hook length.

We regard the random plane partitions (2.5) as the q-deformed random partitions via the expression (2.20) and rewrite the partition function as

$$Z_q^{\mathrm{U}(1)}(Q) = \sum_{p\in\mathbb{Z}} \sum_{\lambda^{(1)},\lambda^{(2)}} Q^{\left|\mu(\lambda^{(1)},\lambda^{(2)};\,p)\right|} s_{\mu(\lambda^{(1)},\lambda^{(2)};\,p)}(q^{-\rho})^2 \,. \tag{3.9}$$

Subsequently we will further factor the above into the following form.

$$Z_q^{\mathrm{U}(1)}(Q) = \sum_{p\in\mathbb{Z}} Z_q^{\mathrm{pert}}(p, Q) \sum_{\lambda^{(1)},\lambda^{(2)}} Z_q^{\mathrm{inst}}(\lambda^{(1)}, \lambda^{(2)}, p,\, Q) \,. \tag{3.10}$$

3.2.1 SU(2) ground partitions

We first consider the cases consisting of the empty partitions

$$\lambda^{(1)} = \lambda^{(2)} = \emptyset \,. \tag{3.11}$$

For each $p \in \mathbb{Z}$, the corresponding partition $\mu(\emptyset, \emptyset;\, p)$ is named the SU(2) ground partition. We will show in the next section that these partitions are responsible to the perturbative gauge theory. It follows from (3.7) that they are given by

$$\mu(\emptyset, \emptyset, p) = \begin{cases} (2p-1, 2p-2, \ldots, 1) & p \geq 1 \\ \emptyset & p = 0 \\ (2|p|, 2|p|-1, \ldots, 1) & p \leq -1 \,. \end{cases} \tag{3.12}$$

The ground partition with $p \geq 1$ is drawn in figure 4.

Taking account of the contributions of the ground partitions to (3.9), we compute the following quantities.

$$\mathcal{Z}_q^{\mathrm{pert}}(p, Q) \equiv Q^{\frac{1}{2}|\mu(\emptyset,\emptyset;\,p)|} s_{\mu(\emptyset,\emptyset;\,p)}(q^{-\rho}) \,. \tag{3.13}$$

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The hook length formula (2.21) becomes useful for the computation. By plugging the hook length of the boxes (figure 4) into the formula, we evaluate (3.13) as follows.

$$
\begin{aligned}
Q^{\frac{1}{2}|\mu(\emptyset,\emptyset;p)|}s_{\mu(\emptyset,\emptyset;p)}(q^{-\rho}) &= Q^{\frac{p(2p-1)}{2}} \prod_{(i,j)\in\mu(\emptyset,\emptyset;p)} \frac{1}{q^{-\frac{h(i,j)}{2}} - q^{\frac{h(i,j)}{2}}} \\
&= \prod_{k=1}^{2p-1} \left\{ \frac{Q^{\frac{1}{2}}}{q^{-\frac{4p-1}{2}}q^{k} - q^{\frac{4p-1}{2}}q^{-k}} \right\}^{k}, \qquad (3.14)
\end{aligned}
$$

for $p \geq 1$. The similar expression is also obtainable for $p \leq -1$. We thus get

$$
\mathcal{Z}_q^{\mathrm{pert}}(p,Q) = \begin{cases} \prod_{k=1}^{2p-1} \left\{ \frac{Q^{\frac{1}{2}}}{Q_F^{-\frac{1}{2}}q^{k} - Q_F^{\frac{1}{2}}q^{-k}} \right\}^{k} & p \geq 1 \\ 1 & p = 0 \\ \prod_{k=1}^{2|p|} \left\{ \frac{Q^{\frac{1}{2}}}{Q_F^{\frac{1}{2}}q^{k} - Q_F^{-\frac{1}{2}}q^{-k}} \right\}^{k} & p \leq -1 \end{cases}, \qquad (3.15)
$$

where we put $Q_F = q^{4p-1}$.

3.2.2 SU(2) instantons

We factorize the partition function (3.9) in the following form.

$$
Z_q^{\mathrm{U}(1)}(Q) = \sum_{p\in\mathbb{Z}} \mathcal{Z}_q^{\mathrm{pert}}(p,Q)^2 \sum_{\lambda^{(1)},\lambda^{(2)}} \mathcal{Z}_q^{\mathrm{inst}}(\lambda^{(1)},\lambda^{(2)},p,\,Q)^2\,, \qquad (3.16)
$$

where we define $\mathcal{Z}_q^{\mathrm{inst}}$ via the relations

$$
Q^{\frac{1}{2}\left|\mu(\lambda^{(1)},\lambda^{(2)};p)\right|}s_{\mu(\lambda^{(1)},\lambda^{(2)};p)}(q^{-\rho}) = \mathcal{Z}_q^{\mathrm{pert}}(p,Q)\mathcal{Z}_q^{\mathrm{inst}}(\lambda^{(1)},\lambda^{(2)},p,\,Q)\,. \qquad (3.17)
$$

Owing to (3.13) we can rewrite the above as

$$
\mathcal{Z}_q^{\mathrm{inst}}(\lambda^{(1)},\lambda^{(2)},p,Q) = Q^{\frac{1}{2}\left(|\mu(\lambda^{(1)},\lambda^{(2)};p)|-|\mu(\emptyset,\emptyset;p)|\right)} \frac{s_{\mu(\lambda^{(1)},\lambda^{(2)};p)}(q^{-\rho})}{s_{\mu(\emptyset,\emptyset;p)}(q^{-\rho})}\,. \qquad (3.18)
$$

The ratio of the Schur functions in (3.18) can be computed by using the infinite product formula of the (specialized) Schur function [15]

$$
s_{\mu}(q^{-\rho}) = q^{-\frac{1}{4}\kappa(\mu)} \prod_{1\leq i<j<\infty} \left\{ \frac{q^{-\frac{1}{2}(x_i(\mu)-x_j(\mu))} - q^{\frac{1}{2}(x_i(\mu)-x_j(\mu))}}{q^{-\frac{1}{2}(j-i)} - q^{\frac{1}{2}(j-i)}} \right\}. \qquad (3.19)
$$

The above formula gives rise to the following expression for $\mathcal{Z}_q^{\mathrm{inst}}$.

$$
\begin{aligned}
&\mathcal{Z}_q^{\mathrm{inst}}(\lambda^{(1)},\lambda^{(2)},p,\,Q) \qquad (3.20) \\
&\quad = Q^{\frac{1}{2}\left(|\mu(\lambda^{(1)},\lambda^{(2)};p)|-|\mu(\emptyset,\emptyset;p)|\right)} q^{-\frac{1}{4}\left(\kappa\left(\mu(\lambda^{(1)},\lambda^{(2)};p)\right)-\kappa(\mu(\emptyset,\emptyset;p))\right)} \times
\end{aligned}
$$

$$\times \prod_{1\le i<j<\infty} \left\{ \frac{q^{-\frac{1}{2}(x_i(\mu(\lambda^{(1)},\lambda^{(2)};p))-x_j(\mu(\lambda^{(1)},\lambda^{(2)};p)))} - q^{\frac{1}{2}(x_i(\mu(\lambda^{(1)},\lambda^{(2)};p))-x_j(\mu(\lambda^{(1)},\lambda^{(2)};p)))}}{q^{-\frac{1}{2}(x_i(\mu(\emptyset,\emptyset;p))-x_j(\mu(\emptyset,\emptyset;p)))} - q^{\frac{1}{2}(x_i(\mu(\emptyset,\emptyset;p))-x_j(\mu(\emptyset,\emptyset;p)))}} \right\}.$$

This is an expression in terms of $\mu(\lambda^{(1)}, \lambda^{(2)}; p)$ and $\mu(\emptyset, \emptyset; p)$. By using (3.7) we can translate (3.20) into an expression presented explicitly in terms of $\lambda^{(1)}$ and $\lambda^{(2)}$. We first note the simple powers of Q and q in (3.20) is read from (3.8) as

$$\begin{aligned} & Q^{\frac{1}{2}\left(|\mu(\lambda^{(1)},\lambda^{(2)};p)|-|\mu(\emptyset,\emptyset;p)|\right)} q^{-\frac{1}{4}\left(\kappa\left(\mu(\lambda^{(1)},\lambda^{(2)};p)\right)-\kappa(\mu(\emptyset,\emptyset;p))\right)} = \\ & = Q^{|\lambda^{(1)}|+|\lambda^{(2)}|} q^{-\kappa(\lambda^{(1)})-\kappa(\lambda^{(2)})} Q_F^{-\frac{|\lambda^{(1)}|-|\lambda^{(2)}|}{2}} . \end{aligned} \tag{3.21}$$

Due to the correspondence (3.7) the elements $x_i(\mu(\lambda^{(1)}, \lambda^{(2)}; p))$ and $x_j(\mu(\emptyset, \emptyset; p))$ are expressed by $x_i(\lambda^{(1)})$, $x_j(\lambda^{(2)})$, $x_k(\emptyset)$ and p. In particular, the products in (3.20) can be classified into the three patterns according as they are made of $q^{-\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)} - q^{x_i(\lambda^{(r)})-x_j(\lambda^{(r)})}$ $(r = 1, 2)$ or $Q_F^{-\frac{1}{2}} q^{-\left(x_i(\lambda^{(1)})-x_j(\lambda^{(2)})\right)} - Q_F^{\frac{1}{2}} q^{x_i(\lambda^{(1)})-x_j(\lambda^{(2)})}$. We then factor the infinite products into these patterns. Putting (3.21) together we get

$$\begin{aligned} \mathcal{Z}_q^{\rm inst}(\lambda^{(1)}, \lambda^{(2)}, p, Q) = & \pm Q^{|\lambda^{(1)}|+|\lambda^{(2)}|} q^{-\kappa(\lambda^{(1)})-\kappa(\lambda^{(2)})} Q_F^{-\frac{|\lambda^{(1)}|-|\lambda^{(2)}|}{2}} \times \\ & \times \prod_{r=1,2} \left[\prod_{1\le i<j<\infty} \left\{ \frac{q^{-\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)} - q^{x_i(\lambda^{(r)})-x_j(\lambda^{(r)})}}{q^{-(j-i)} - q^{j-i}} \right\} \right] \times \\ & \times \prod_{1\le i,j<\infty} \left\{ \frac{Q_F^{-\frac{1}{2}} q^{-\left(x_i(\lambda^{(1)})-x_j(\lambda^{(2)})\right)} - Q_F^{\frac{1}{2}} q^{x_i(\lambda^{(1)})-x_j(\lambda^{(2)})}}{Q_F^{-\frac{1}{2}} q^{-(j-i)} - Q_F^{\frac{1}{2}} q^{j-i}} \right\} \end{aligned} \tag{3.22}$$

The expression (3.22) makes it possible to rewrite $\mathcal{Z}_q^{\rm inst}$ in a form convenient to compare with the topological string amplitude. The first two infinite products have the same form. By noting the relation $s_\mu(q^{-\rho}) = (-)^{|\mu|} q^{-\frac{1}{2}\kappa(\mu)} s_\mu(q^\rho)$, the product formula (3.19) gives

$$\prod_{1\le i<j<\infty} \left\{ \frac{q^{-\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)} - q^{x_i(\lambda^{(r)})-x_j(\lambda^{(r)})}}{q^{-(j-i)} - q^{j-i}} \right\} = (-)^{|\lambda^{(r)}|} q^{-\frac{1}{2}\kappa(\lambda^{(r)})} s_{\lambda^{(r)}}(q^{2\rho}) . \tag{3.23}$$

As regards the last infinite products in (3.22), the following description is obtainable by the standard computation of $2d$ CFT.

$$\begin{aligned} & \prod_{1\le i,j<\infty} \left\{ \frac{Q_F^{-\frac{1}{2}} q^{-(x_i(\lambda^{(1)})-x_j(\lambda^{(2)}))} - Q_F^{\frac{1}{2}} q^{x_i(\lambda^{(1)})-x_j(\lambda^{(2)})}}{Q_F^{-\frac{1}{2}} q^{-(j-i)} - Q_F^{\frac{1}{2}} q^{j-i}} \right\} = \\ & = q^{\frac{1}{2}(\kappa(\lambda^{(1)})-\kappa(\lambda^{(2)}))} Q_F^{\frac{1}{2}(|\lambda^{(1)}|+|\lambda^{(2)}|)} \prod_{k=1}^{\infty} (1 - Q_F q^{2k})^k \sum_\nu Q_F^{|\nu|} s_\nu(q^{2(\lambda^{(1)}+\rho)}) s_\nu(q^{2(\widetilde{\lambda^{(2)}}+\rho)}) , \end{aligned} \tag{3.24}$$

where $\widetilde{\lambda}$ denotes the partition conjugate to λ. Finally, putting together (3.23) and (3.24), we can rewrite (3.22) as follows;

$$\begin{aligned} \mathcal{Z}_q^{\rm inst}(\lambda^{(1)}, \lambda^{(2)}, p, Q) = & \pm Q^{|\lambda^{(1)}|+|\lambda^{(2)}|} q^{-\kappa(\lambda^{(1)})-2\kappa(\lambda^{(2)})} Q_F^{|\lambda^{(2)}|} s_{\lambda^{(1)}}(q^{2\rho}) s_{\lambda^{(2)}}(q^{2\rho}) \times \\ & \times \prod_{k=1}^{\infty} (1 - Q_F q^{2k})^k \sum_\nu Q_F^{|\nu|} s_\nu(q^{2(\lambda^{(1)}+\rho)}) s_\nu(q^{2(\widetilde{\lambda^{(2)}}+\rho)}) . \end{aligned} \tag{3.25}$$

3.2.3 Interpretation as five-dimensional SU(2) Yang-Mills

We fix $p \in \mathbb{Z}$. The relevant field theory parameters are a, Λ and R, where $\pm a$ are the VEVs of the adjoint scalar in the vector multiplet. We identify q, Q_F and Q with a, Λ and R as follows.

$$q = e^{-R\hbar}\,, \qquad Q_F = e^{-4Ra}\,, \qquad Q = -(2R\Lambda)^2\,. \tag{3.26}$$

Since we have set $Q_F = q^{4p-1}$ the above implies

$$a = \hbar\widetilde{p}\,, \tag{3.27}$$

where $\widetilde{p} \equiv p - \frac{1}{4}$.

The identifications (3.26) convert $\mathcal{Z}_q^{\mathrm{inst}}$ to the instanton contributions in five-dimensional gauge theories. This follows by rephrasing the expression (3.22) in terms of the field theory parameters. In particular, we obtain

$$\sum_{\lambda^{(1)},\lambda^{(2)}} \mathcal{Z}_q^{\mathrm{inst}}(\lambda^{(1)}, \lambda^{(2)}, p, Q)^2 = Z_{5d\,\mathrm{SYM}}^{\mathrm{SU(2)\,inst}}(a;\, \Lambda, R, \hbar)\,. \tag{3.28}$$

The r.h.s. is the instanton part of the exact partition function [2] for five-dimensional $\mathcal{N}=1$ supersymmetric SU(2) Yang-Mills with the Chern-Simons term [13]. The Chern-Simons corrections come from the following factor in (3.22).

$$q^{-\kappa(\lambda^{(1)})-\kappa(\lambda^{(2)})} Q_F^{-\frac{|\lambda^{(1)}|-|\lambda^{(2)}|}{2}}\,, \tag{3.29}$$

which is also understood [17] as a part of the so-called framing factor of the topological string vertices [6]. The Chern-Simons term is quantized with the coupling constant c_{cs} taking integral values. In the present case, it becomes $c_{cs} = 2$.

In the next section we will show that the square of $\mathcal{Z}_q^{\mathrm{pert}}$ is translated to the perturbative part of the partition function for the supersymmetric Yang-Mills. Taking it for granted for a while, together with (3.28) we find

$$\mathcal{Z}_q^{\mathrm{pert}}(p, Q)^2 \sum_{\lambda^{(1)},\lambda^{(2)}} \mathcal{Z}_q^{\mathrm{inst}}(\lambda^{(1)}, \lambda^{(2)}, p, Q)^2 = Z_{5d\,\mathrm{SYM}}^{\mathrm{SU(2)}}(a;\, \Lambda, R, \hbar)\,. \tag{3.30}$$

The r.h.s. is the exact partition function [2] for the five-dimensional supersymmetric SU(2) Yang-Mills with the Chern-Simons term. We notice that, as confirmed [2] for four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories, the gauge theory partition function is realized as the $\hbar \to 0$ limit of the above partition function.

It is clear from (3.30) that the field theory limit is the thermodynamic limit with a, Λ and R fixed. The identification (3.27) makes $p = o(\hbar^{-1})$. The corresponding ground partition in (3.30) becomes very large. The partitions $\lambda^{(r)}$ represent U(1) gauge instantons with $c_2 = |\lambda^{(r)}|$ on a non-commutative $\mathbb{R}^4$. In the model of random partitions, it follows from (3.7) that these gauge instantons become excitations from the ground partition. They provide the deformation of the ground partition at the thermodynamic limit (figure 5). These imply that the ground partitions describe the perturbative regime of the Coloumb branch while their deformation becomes non-perturbative in the gauge theory.

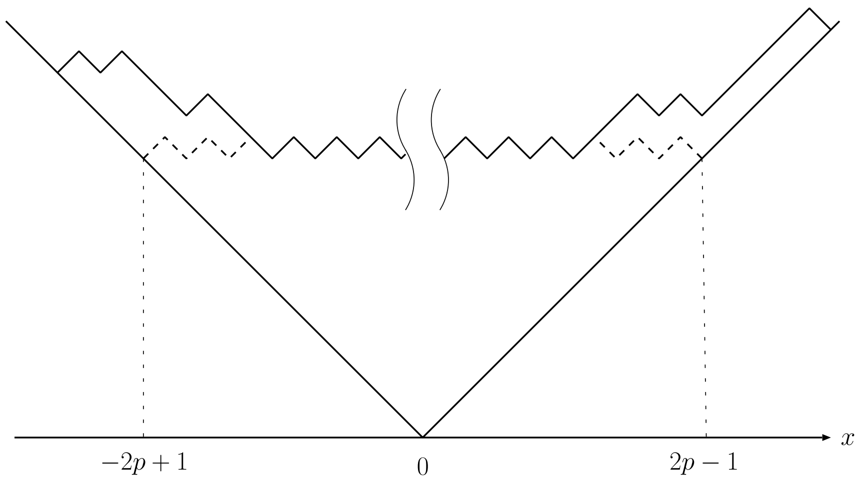


Figure 5: Deformation of the SU(2) ground partition for $p \gg 1$ by two partitions $(3, 1, 1)$ and $(3, 2)$.

3.3 Interpretation as topological string amplitude

The relevant non-compact Calabi-Yau threefold is the local $\mathbb{F}_2$. The geometric engineering [5] dictates that four-dimensional $\mathcal{N} = 2$ supersymmetric SU(2) Yang-Mills is realized by the canonical bundle over the Hirzebruch surface $\mathbb{F}_m$. It is an ALE space with A_1 singularity fibred over $\mathbb{P}^1$. Type of the fibrations of the ALE space is labelled by the integer m, which is called the framing and taking values $0, 1, 2$. We choose the framing to be $m = 2$. The geometrical data are the Kähler volumes of the compact two-cycles. We denote the Kähler parameters of the base $\mathbb{P}^1$ and the fibre $\mathbb{P}^1$ (the blow-up cycle of A_1 singularity) respectively by t_B and t_F.

Topological string amplitudes on the geometries dictated by the geometric engineering are computed in [7, 8] by using the topological vertices. We summarize these amplitudes in appendix. Since they are generating functions of the world-sheet instantons, string coupling constant g_{st} and the Kähler parameters appear in the amplitudes as certain combinations of $q = e^{-g_{st}}$,$Q_B = e^{-t_B}$ and $Q_F = e^{-t_F}$.

Let us denote the topological string amplitude on the local $\mathbb{F}_2$ by $Z^{\mathrm{SU}(2)}_{\mathrm{string}}(q, Q_F, Q_B)$. It is given in (A.3). By comparing (3.25) with (A.3), we see that the gauge instanton contribution (3.28) can be converted to the string amplitude as follows.

$$\sum_{\lambda^{(1)},\lambda^{(2)}} \mathcal{Z}^{\mathrm{inst}}_q(\lambda^{(1)}, \lambda^{(2)}, p, Q)^2 = \prod_{k=1}^{\infty} (1 - Q_F q^{2k})^{2k}\, Z^{\mathrm{SU}(2)}_{\mathrm{string}}(q^2, Q_F, Q^2)\,, \tag{3.31}$$

where we put $Q_F = q^{4\widetilde{p}}$. This is in accord with the result [7, 8] that the topological string amplitude on the local $\mathbb{F}_2$ leads to the instanton part of the partition function of five-dimensional supersymmetric SU(2) Yang-Mills.

Due to the identification $\hbar \sim g_{st}$, the thermodynamic limit is in the perturbative regime of string theory. The previous consideration about the role of the ground partitions in gauge theory could be translated in string theory. The ground partitions at the thermodynamic limit must be interpreted as classical objects in string theory. The relation $t_F = 4Ra$, which follows from (3.31), suggests that the ground partitions describe the resolution of A_1 singularity in the local $\mathbb{F}_2$.

4. Perturbative gauge theory from ground partitions

The aim of this section is to show that the square of $\mathcal{Z}_q^{\mathrm{pert}}(p,Q)$ in (3.30) describes the perturbative part of the partition function for the five-dimensional supersymmetric Yang-Mills.

Let $\mathcal{G}_n$ be the partition $(n, n-1, \cdots, 1)$, where n is a positive integer. The ground partition $\mu(\emptyset,\emptyset;p)$ for $p \in \mathbb{Z}_{\geq 1}$ is $\mathcal{G}_{2p-1}$.[1] Let us consider the logarithm of $\mathcal{Z}_q^{\mathrm{pert}}(p,Q)^2$. It can be written as follows.

$$\begin{aligned} -\ln \mathcal{Z}_q^{\mathrm{pert}}(p,Q)^2 &= \ln \prod_{(i,j)\in\mathcal{G}_{2p-1}} \frac{\left(q^{-\frac{h(i,j)}{2}} - q^{\frac{h(i,j)}{2}}\right)^2}{Q} \\ &= \ln \prod_{k=1}^{2p-1} \frac{\left(e^{R\hbar(2\tilde{p}-k)} - e^{-R\hbar(2\tilde{p}-k)}\right)^{2k}}{(-)^k (2R\Lambda)^{2k}}\,, \end{aligned} \tag{4.1}$$

where q and Q are translated to Λ and R by (3.26). We also note $\widetilde{p} = p - \frac{1}{4}$. We further factor the logarithm as

$$\begin{aligned} -\ln \mathcal{Z}_q^{\mathrm{pert}}(p,Q)^2 &= \sum_{k=1}^{2p-1} k \ln\left(\frac{e^{R\hbar(2\tilde{p}-k)} - e^{-R\hbar(2\tilde{p}-k)}}{2R\Lambda}\right) + \sum_{k=1}^{2p-1} k \ln\left(\frac{e^{-R\hbar(2\tilde{p}-k)} - e^{R\hbar(2\tilde{p}-k)}}{2R\Lambda}\right) \\ &= g(2p\,|\hbar, 2R, \Lambda) + g(2p\,|-\hbar, 2R, \Lambda)\,, \end{aligned} \tag{4.2}$$

where we introduce

$$g(2p\,|\hbar, 2R, \Lambda) = \sum_{k=1}^{2p-1} k \ln\left(\frac{\sinh R\hbar(2\tilde{p}-k)}{R\Lambda}\right). \tag{4.3}$$

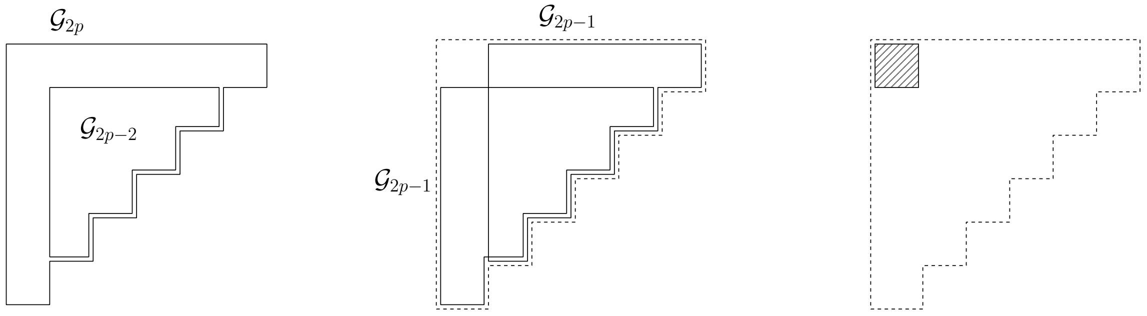
The function (4.3) becomes identical to the function $\gamma_\hbar(x;\beta;\Lambda)$ given in [2]. To confirm this, it should be tested first of all by the q-difference equation [2].

Proposition 1. *The function (4.3) satisfies the q-difference equation*

$$g(2p+1|\hbar, 2R, \Lambda) + g(2p-1|\hbar, 2R, \Lambda) - 2g(2p|\hbar, 2R, \Lambda) = \ln\left(\frac{\sinh R(2\hbar\tilde{p})}{R\Lambda}\right). \tag{4.4}$$

Let us verify (4.4) by using a recurrence relation among the Young diagrams $\mathcal{G}_n$. The Young diagrams $\mathcal{G}_{2p}$ and $\mathcal{G}_{2p-2}$ are piled in figure 6a so that any two boxes at a common position have the same hook length. While in figure 6b, two $\mathcal{G}_{2p-1}$ are slightly shifted to

[1] The ground partition for $p \in \mathbb{Z}_{\leq -1}$ is $\mathcal{G}_{|2p|}$. The discussion presented below goes as well in these cases.



(a) (b) (c)

Figure 6: The Young diagrams for the q-difference equation.

cover the piled diagrams of (a). Each box of the diagrams has the same hook length as that of $\mathcal{G}_{2p}$ at the same position. By noting (4.1), we exponentiate the l.h.s. of (4.4) to

$$\prod_{(i,j)\in\mathcal{G}_{2p}} (q^{-\frac{h(i,j)}{2}} - q^{\frac{h(i,j)}{2}}) \prod_{(i,j)\in\mathcal{G}_{2p-1}} (q^{-\frac{h(i,j)}{2}} - q^{\frac{h(i,j)}{2}})^{-1} \times \\ \times \prod_{(i,j)\in\mathcal{G}_{2p-1}} (q^{-\frac{h(i,j)}{2}} - q^{\frac{h(i,j)}{2}})^{-1} \prod_{(i,j)\in\mathcal{G}_{2p-2}} (q^{-\frac{h(i,j)}{2}} - q^{\frac{h(i,j)}{2}})\,. \qquad (4.5)$$

It follows from the above explanation of the figures that nearly all the products which come from $\mathcal{G}_{2p}$ and $\mathcal{G}_{2p-2}$ in (4.5) cancel with the products coming from two $\mathcal{G}_{2p-1}$, and that the net becomes only the contribution from the shaded box in figure 6c. Hence, (4.5) is equal to $q^{-\frac{4p-1}{2}} - q^{\frac{4p-1}{2}}$. This turns to give the r.h.s. of (4.4).

Proposition 2. *The function $g(2p|\hbar, 2R, \Lambda)$ coincides with the function $\gamma_\hbar(2\hbar\tilde{p}|2R;\Lambda)$ in [2], up to a linear function in $\hbar\tilde{p}$.*

To prove the above we rewrite (4.3) as

$$g(2p|\hbar, 2R, \Lambda) = -\sum_{k=1}^{2p-1} k\ln(2R\Lambda) + R\hbar\sum_{k=1}^{2p-1} k(2\tilde{p}-k) + \sum_{k=1}^{2p-1} k\ln\left(1 - e^{-2R\hbar(2\tilde{p}-k)}\right). \quad (4.6)$$

The first two terms become

$$\begin{aligned} -\sum_{k=1}^{2p-1} k\ln(2R\Lambda) &= -\frac{(2\hbar\widetilde{p})^2}{2\hbar^2}\ln(2R\Lambda) + \frac{1}{8}\ln(2R\Lambda)\,, \\ R\hbar\sum_{k=1}^{2p-1} k(2\widetilde{p}-k) &= \frac{R(2\hbar\widetilde{p})^3}{6\hbar^2} - \frac{R(2\hbar\widetilde{p})}{24}\,. \end{aligned} \qquad (4.7)$$

As regards the last term of (4.6) we rewrite the finite sum as a subtraction between two infinite sums and then expand the logarithms

$$\sum_{k=1}^{2p-1} k\ln\left(1 - e^{-2R\hbar(2\tilde{p}-k)}\right) = \left\{\sum_{k=1}^{+\infty} - \sum_{k=2p}^{+\infty}\right\} k\ln\left(1 - e^{-2R\hbar(2\tilde{p}-k)}\right) \qquad (4.8)$$

$$= -\sum_{k=1}^{\infty} k \sum_{m=1}^{\infty} \frac{1}{m} e^{-2Rm\hbar(2\tilde{p}-k)} + \sum_{k=2p}^{\infty} k \sum_{m=1}^{\infty} \frac{1}{m} e^{-2Rm\hbar(2\tilde{p}-k)} .$$

The summations over k in the above give rise to

$$\sum_{k=1}^{2p-1} k \ln(1 - e^{-2Rm\hbar(2\tilde{p}-k)}) = \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-Rm(2\hbar\tilde{p})}}{(1-e^{2Rm\hbar})(1-e^{-2Rm\hbar})} + \left(\tilde{p} + \frac{1}{4}\right) \times \tag{4.9}$$
$$\times \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{Rm\hbar}}{1-e^{2Rm\hbar}} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{Rm\hbar}}{(1-e^{2Rm\hbar})(1-e^{-2Rm\hbar})} ,$$

where the first term provides an analogue of the McMahon function, and the last two terms are a linear function in $\hbar\tilde{p}$.

Collecting (4.7) and (4.9), we obtain the following expression for $g(2p|\hbar, 2R, \Lambda)$.

$$\begin{aligned} g(2p|\hbar, 2R, \Lambda) &= \alpha(\hbar, 2R, \Lambda) + \beta(\hbar, 2R, \Lambda)(\hbar\tilde{p}) - \\ &\quad - \frac{(2\hbar\tilde{p})^2}{2\hbar^2} \ln(2R\Lambda) + \frac{R(2\hbar\tilde{p})^3}{6\hbar^2} + \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-2Rm(2\hbar\tilde{p})}}{(1-e^{2Rm\hbar})(1-e^{-2Rm\hbar})} \\ &= \alpha(\hbar, 2R, \Lambda) + \beta(\hbar, 2R, \Lambda)(\hbar\tilde{p}) + \gamma_\hbar(2\hbar\tilde{p}|2R, \Lambda) . \end{aligned} \tag{4.10}$$

Thus up to a linear function in $\hbar\tilde{p}$, the function (4.3) is identified with the perturbative term given in [2]. We lastly remark that $g(p|\hbar, 2R, \Lambda)$ has the smooth $R \to 0$ limit, as is obvious from (4.3).

By the same argument as above we can also express $g(2p|-\hbar, 2R, \Lambda)$ in a form analogous to (4.10). It becomes as follows.

$$\begin{aligned} g(2p| - \hbar, 2R, \Lambda) &= \alpha(-\hbar, 2R, \Lambda) + \beta(-\hbar, 2R, \Lambda)(-\hbar\tilde{p}) - \\ &\quad - \frac{(2\hbar\tilde{p})^2}{2\hbar^2} \ln(2R\Lambda) - \frac{R(2\hbar\tilde{p})^3}{6\hbar^2} + \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{2Rm(2\hbar\tilde{p})}}{(1-e^{2Rm\hbar})(1-e^{-2Rm\hbar})} \\ &= \alpha(-\hbar, 2R, \Lambda) + \beta(-\hbar, 2R, \Lambda)(-\hbar\tilde{p}) + \gamma_h(-2\hbar\tilde{p}|R, \Lambda) . \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we arrive at

$$\begin{aligned} \mathcal{Z}_q^{\rm pert}(p, Q)^2 &= \exp\Big\{ -g(2p|\hbar, 2R, \Lambda) - g(2p| - \hbar, 2R, \Lambda) \Big\} \\ &= \exp\Big\{ -\Delta^{(0)}(\hbar, 2R, \Lambda) - \Delta^{(1)}(\hbar, 2R, \Lambda)(\hbar\widetilde{p}) \Big\} \times \\ &\quad \times \exp\left\{ - \Big(\gamma_\hbar(2\hbar\tilde{p}|2R, \Lambda) + \gamma_\hbar(-2\hbar\tilde{p}|2R, \Lambda) \Big) \right\} , \end{aligned} \tag{4.12}$$

where $\Delta^{(0)}$ and $\Delta^{(1)}$ are the collections of α and β in (4.10) and (4.11). In terms of the VEV of the adjoint scalar (3.27) the above becomes

$$\begin{aligned} \mathcal{Z}_q^{\rm pert}(p, Q)^2 &= \exp\Big\{ -\Delta^{(0)}(\hbar, 2R, \Lambda) - \Delta^{(1)}(\hbar, 2R, \Lambda)a \Big\} \times \\ &\quad \times \exp\left\{ - \Big(\gamma_\hbar(2a|2R, \Lambda) + \gamma_\hbar(-2a|2R, \Lambda) \Big) \right\} . \end{aligned} \tag{4.13}$$

The expression (4.13) shows that the square of $\mathcal{Z}_q^{\rm pert}$ can be identified with the perturbative part of the partition function for the five-dimensional supersymmetric Yang-Mills.

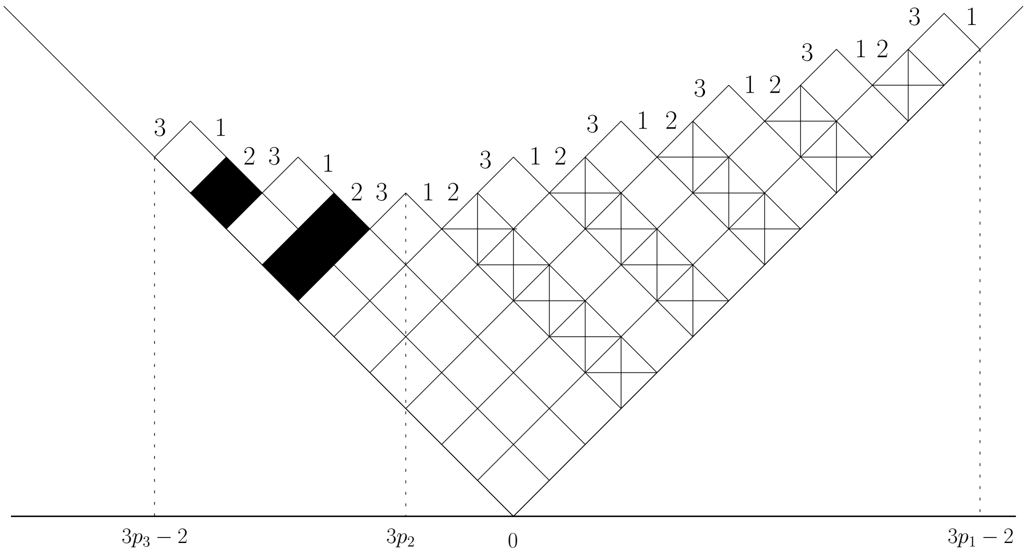


Figure 7: The SU(3) ground partition for $(p_1, p_2, p_3) = (5, -1, -4)$. The rows and the columns are coloured by $1, 2$ and 3. The Young diagram is bicoloured by $(1, 2), (1, 3)$ and $(2, 3)$. The corresponding boxes are denoted respectively by ⊠, □ and ■.

5. SU(N) Yang-Mills and random plane partitions

The previous discussions on the SU(2) gauge theory could be generalized to the higher rank gauge groups. Owing to the identification (3.3) of N-component fermions any charged partition (μ, n) can be expressed by a set of N charged partitions $(\lambda^{(r)}, p_r)$. We consider in the neutral sector, that is, $n = \sum_{r=1}^{N} p_r = 0$. $(\lambda^{(r)}, p_r)$ determine a partition $\mu\left(\{\lambda^{(r)}\};\,\{p_r\}\right)$ in this sector. The corresponding Maya diagram $S_{\mu\left(\{\lambda^{(r)}\};\,\{p_r\}\right)}$ can be read from (3.5) as

$$S_{\mu\left(\{\lambda^{(r)}\};\,\{p_r\}\right)} = \bigcup_{r=1}^{N} \left\{ N(x_{i_r}(\lambda^{(r)}) + \widetilde{p}_r)\ ; i_r \geq 1 \right\}. \tag{5.1}$$

In this setting it is convenient to colour-code a partition. We N-colour the upper edges of the Young diagram by attaching r to the edge when the projection of the middle point belongs to $N\mathbb{Z} + r - \frac{1}{2}$. Thereby rows and columns of the Young diagram are N-coloured. See figure 7 for the case of $N = 3$. The box (i, j) in the Young diagram is bicoloured by (r_i, s_j), where r_i and s_j are respectively the colors of the i-th row and the j-th column. The box bicoloured by (r, s) is simply called (r, s)-box.

We regard the random plane partitions (2.5) as the q-deformed random partitions via (2.20) and rewrite the partition function as

$$Z_q^{\mathrm{U}(1)}(Q) = \sum_{\{p_r\}} \sum_{\{\lambda^{(r)}\}} Q^{\left|\mu\left(\{\lambda^{(r)}\};\,\{p_r\}\right)\right|} s_{\mu\left(\{\lambda^{(r)}\};\,\{p_r\}\right)}(q^{-\rho})^2\,. \tag{5.2}$$

Similarly to the case of SU(2), we will further factor the above into the following form.

$$Z_q^{\mathrm{U}(1)}(Q) = \sum_{\{p_r\}} Z_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\}, Q) \sum_{\{\lambda^{(r)}\}} Z_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\}, \{p_r\}, Q)\,. \tag{5.3}$$

SU(N) **ground partitions.** We consider the cases of the empty partitions

$$\lambda^{(r)} = \emptyset \qquad (1 \leq \forall r \leq N)\,. \tag{5.4}$$

For each $\{p_r\}$, the corresponding partition $\mu(\{\emptyset\};\{p_r\})$ is named the SU(N) ground partition. The case of $N=3$ is illustrated in figure 7. Their contributions to the partition function lead us to introduce $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\},Q)$ by

$$q^{-\frac{N^2}{12}\sum_{s=1}^N \widetilde{p}_s{}^3}\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\},Q) \equiv Q^{\frac{1}{2}|\mu(\{\emptyset\};\,\{p_r\})|}s_{\mu(\{\emptyset\};\,\{p_r\})}(q^{-\rho})\,. \tag{5.5}$$

In other words, we put

$$\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\},Q) = \prod_{(i,j)\in\mu(\{\emptyset\};\,\{p_r\})}\frac{Q^{\frac{1}{2}}}{q^{-\frac{h(i,j)}{2}}-q^{\frac{h(i,j)}{2}}}\,. \tag{5.6}$$

This follows from (5.5) by applying the formula (2.21) and noting[2] $\kappa(\mu(\{\emptyset\};\{p_r\})) = \frac{N^2}{3}\sum_{s=1}^N \widetilde{p}_s^3$.

Let us compute $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}$ given in the above. It is instructive to start with the case of $N=3$. To simplify the discussion, we order the three U(1) charges $p_{1,2,3}$ as $p_1>p_2>p_3$. They satisfy $p_1+p_2+p_3=0$ by the neutral condition. The ground partition for (p_1,p_2,p_3) is bicoloured by $(1,2),(1,3)$ and $(2,3)$. We factor the products in (5.6) according to the color-coding of the partition as follows.

$$\begin{aligned}\prod_{(i,j)\in\mu(\{\emptyset\};\{p_1,p_2,p_3\})}\left(q^{-\frac{h(i,j)}{2}}-q^{\frac{h(i,j)}{2}}\right)^{-1} &= \prod_{\boxtimes\in\mu(\{\emptyset\};\{p_1,p_2,p_3\})}\left(q^{-\frac{h(\boxtimes)}{2}}-q^{\frac{h(\boxtimes)}{2}}\right)^{-1}\times\\ &\quad\times\prod_{\square\in\mu(\{\emptyset\};\{p_1,p_2,p_3\})}\left(q^{-\frac{h(\square)}{2}}-q^{\frac{h(\square)}{2}}\right)^{-1}\times\\ &\quad\times\prod_{\blacksquare\in\mu(\{\emptyset\};\{p_1,p_2,p_3\})}\left(q^{-\frac{h(\blacksquare)}{2}}-q^{\frac{h(\blacksquare)}{2}}\right)^{-1},\end{aligned} \tag{5.7}$$

where we depict $(1,2)$-box as $\boxtimes$, $(1,3)$-box as $\square$ and $(2,3)$-box as $\blacksquare$. Each term of (5.7) can be computed so that it acquires a form analogous to (3.14). For instance, the first term becomes

$$\prod_{\boxtimes\in\mu(\{\emptyset\};\{p_1,p_2,p_3\})}\left(q^{-\frac{h(\boxtimes)}{2}}-q^{\frac{h(\boxtimes)}{2}}\right)^{-1} = \prod_{k=1}^{p_1-p_2-1}\left(q^{-\frac{3}{2}(\widetilde{p}_1-\widetilde{p}_2-k)}-q^{\frac{3}{2}(\widetilde{p}_1-\widetilde{p}_2-k)}\right)^{-k}. \tag{5.8}$$

These give the following expression for $\mathcal{Z}_q^{\mathrm{SU}(3)\,\mathrm{pert}}$;

$$\mathcal{Z}_q^{\mathrm{SU}(3)\,\mathrm{pert}}(p_1,p_2,p_3,Q) = \prod_{1\leq r<s\leq 3}\prod_{k=1}^{p_r-p_s-1}\left\{\frac{Q^{\frac{1}{2}}}{q^{-\frac{3}{2}(\widetilde{p}_r-\widetilde{p}_s-k)}-q^{\frac{3}{2}(\widetilde{p}_r-\widetilde{p}_s-k)}}\right\}^k. \tag{5.9}$$

[2] $\kappa(\mu)$ measures asymmetry of the Young diagram.

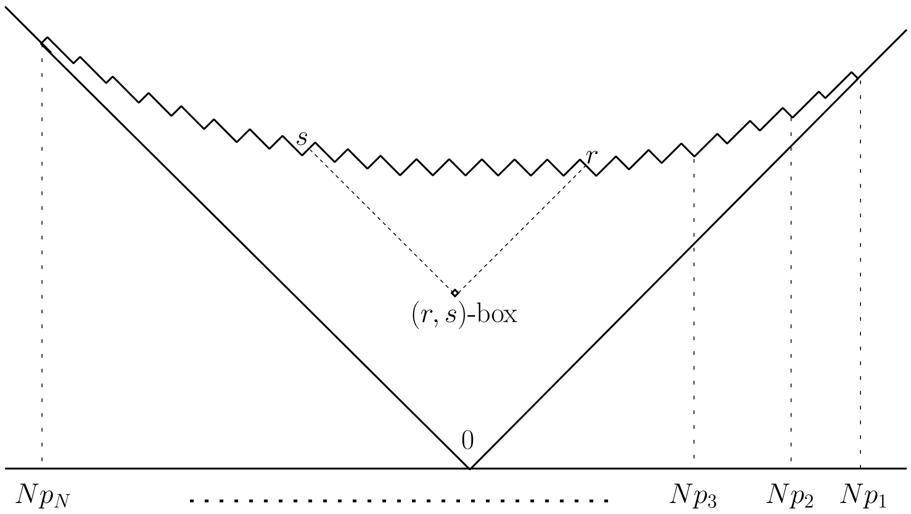


Figure 8: A general view of the SU(N) ground partition.

For the cases of general values of N, we also order the U(1) charges as $p_1 > p_2 > \cdots > p_N$. They satisfy $\sum_{r=1}^{N} p_r = 0$. The SU(N) ground partition for $\{p_r\}$ is now bicoloured by (r, s) with $1 \leq r < s \leq N$. See figure 8 for the illustration. We factor the products in (5.6) according to the color-coding of the partition as in the case of $N = 3$.

$$\prod_{(i,j)\in\mu(\{\emptyset\};\{p_s\})} \left(q^{-\frac{h(i,j)}{2}} - q^{\frac{h(i,j)}{2}}\right)^{-1} = \prod_{1\leq r<s\leq N} \prod_{(r,s)\text{-boxes}} \left(q^{-\frac{h((r,s)\text{-box})}{2}} - q^{\frac{h((r,s)\text{-box})}{2}}\right)^{-1}. \tag{5.10}$$

The contribution of the (r, s)-boxes can be computed in a form similar to (5.8). We thus obtain the following expression for $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}$;

$$\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\}, Q) = \prod_{1\leq r<s\leq N} \prod_{k=1}^{p_r-p_s-1} \left\{ \frac{Q^{\frac{1}{2}}}{\left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{-\frac{1}{2}} q^{\frac{Nk}{2}} - \left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{\frac{1}{2}} q^{-\frac{Nk}{2}}} \right\}^k, \tag{5.11}$$

where we put $Q_{F_r} \equiv q^{N(\widetilde{p}_r - \widetilde{p}_{r+1})}$ for $1 \leq r \leq N$.

SU(N) **instantons.** We factorize the partition function (5.2) in the following form.

$$Z_q^{\mathrm{U}(1)}(Q) = \sum_{\{p_r\}} q^{-\frac{N^2}{6}\sum_{s=1}^{N}\widetilde{p}_s^3}\, \mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\}, Q)^2 \sum_{\{\lambda^{(r)}\}} \mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\}, \{p_r\}, Q)^2, \tag{5.12}$$

where $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}$ are defined by the relations

$$\begin{aligned} Q^{\frac{1}{2}|\mu(\{\lambda^{(r)}\};\{p_r\})|} s_{\mu(\{\lambda^{(r)}\};\{p_r\})}(q^{-\rho}) &= q^{-\frac{N^2}{12}\sum_{s=1}^{N}\widetilde{p}_s{}^3} \times \\ &\times \mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\}, Q)\, \mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\}, \{p_r\}, Q). \end{aligned} \tag{5.13}$$

Owing to (5.5) we can write the above as

$$\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\}, \{p_r\}, Q) = Q^{\frac{1}{2}\left(|\mu(\{\lambda^{(r)}\};\{p_r\})| - |\mu(\{\emptyset\};\{p_r\})|\right)} \frac{s_{\mu(\{\lambda^{(r)}\};\{p_r\})}(q^{-\rho})}{s_{\mu(\{\emptyset\};\{p_r\})}(q^{-\rho})}. \tag{5.14}$$

Let us compute $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}$. We order $p_1 > p_2 > \cdots > p_N$ for simplicity. The ratio of the Schur functions in (5.14) is translated to infinite products by applying the formula (3.19). By using the description (5.1) we can classify the ingredients of the products according as they are made of $q^{-\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)} - q^{\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)}$ where $1 \le r \le N$, or $(\prod_{t=r}^{s-1} Q_{F_t})^{-\frac{1}{2}} q^{-\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(s)})\right)} - (\prod_{t=r}^{s-1} Q_{F_t})^{\frac{1}{2}} q^{\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(s)})\right)}$ where $1 \le r < s \le N$. We then factor the r.h.s. of (5.14) by this classification. Taking (3.6) into account, we obtain the following expression for $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}$.

$$
\begin{aligned}
&\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\},\{p_r\},Q) = \\
&= \pm Q^{\frac{N}{2}\sum_{r=1}^N |\lambda^{(r)}|} q^{-\frac{N^2}{4}\sum_{r=1}^N \kappa(\lambda^{(r)})} \prod_{r<s}^{N} \left(\prod_{t=r}^{s-1} Q_{F_t} \right)^{-\frac{|\lambda^{(r)}|-|\lambda^{(s)}|}{2}} \times \qquad (5.15) \\
&\times \prod_{r=1}^{N} \prod_{1\le i<j<\infty} \left\{ \frac{q^{-\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)} - q^{\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)}}{q^{-\frac{N(j-i)}{2}} - q^{\frac{N(j-i)}{2}}} \right\} \times \\
&\times \prod_{r<s}^{N} \prod_{1\le i,j<\infty} \left\{ \frac{\left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{-\frac{1}{2}} q^{-\frac{N}{2}(x_i(\lambda^{(r)})-x_j(\lambda^{(s)}))} - \left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{\frac{1}{2}} q^{\frac{N}{2}(x_i(\lambda^{(r)})-x_j(\lambda^{(s)}))}}{\left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{-\frac{1}{2}} q^{-\frac{N(j-i)}{2}} - \left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{\frac{1}{2}} q^{\frac{N(j-i)}{2}}} \right\}.
\end{aligned}
$$

The expression (5.15) makes it possible to rewrite $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}$ in a form convenient to compare with the topological string amplitude. We first notice the identity

$$
\prod_{1\le i<j<\infty} \left\{ \frac{q^{-\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)} - q^{\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(r)})\right)}}{q^{-\frac{N(j-i)}{2}} - q^{\frac{N(j-i)}{2}}} \right\} = (-)^{|\lambda^{(r)}|} q^{-\frac{N}{4}\kappa(\lambda^{(r)})} s_{\lambda^{(r)}}(q^{N\rho}), \qquad (5.16)
$$

which follows from the product formula (3.19). The identity (3.24) of the SU(2) theory has the following generalization.

$$
\begin{aligned}
&\prod_{r<s}^{N} \prod_{1\le i,j<\infty} \left\{ \frac{\left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{-\frac{1}{2}} q^{-\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(s)})\right)} - \left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{\frac{1}{2}} q^{\frac{N}{2}\left(x_i(\lambda^{(r)})-x_j(\lambda^{(s)})\right)}}{\left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{-\frac{1}{2}} q^{-\frac{N(j-i)}{2}} - \left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{\frac{1}{2}} q^{\frac{N(j-i)}{2}}} \right\} = \\
&= \prod_{r<s}^{N} \left\{ q^{\frac{N}{4}\left(\kappa(\lambda^{(r)})-\kappa(\lambda^{(s)})\right)} \left(\prod_{t=r}^{s-1} Q_{F_t} \right)^{\frac{1}{2}\left(|\lambda^{(r)}|+|\lambda^{(s)}|\right)} \prod_{k=1}^{\infty} \left(1 - \left(\prod_{t=r}^{s-1} Q_{F_t} \right) q^{Nk} \right)^{k} \right\} \times \\
&\times \sum_{\nu^{(1)},\cdots,\nu^{(N-1)}} \left(\prod_{t=1}^{N-1} Q_{F_t}^{|\nu^{(t)}|} \right) s_{\nu^{(1)}}(q^{N(\lambda^{(1)}+\rho)}) s_{\nu^{(N-1)}}(q^{N(\widetilde{\lambda^{(N)}}+\rho)}) \\
&\times \sum_{\chi^{(1)},\cdots,\chi^{(N-2)}} \prod_{t=1}^{N-2} s_{\nu^{(t)}/\chi^{(t)}}(q^{N(\widetilde{\lambda^{(t)}}+\rho)}) s_{\nu^{(t+1)}/\chi^{(t)}}(q^{N(\lambda^{(t)}+\rho)}) . \qquad (5.17)
\end{aligned}
$$

The substitution of (5.16) and (5.17) into (5.15) gives rise to some simple powers of q and Q_{F_r} in addition to a bunch of the Schur functions. The exponent of q can be read as

$$
q^{-\frac{N^2}{4}\sum_{r=1}^N \kappa(\lambda^{(r)})} \cdot q^{-\frac{N}{4}\sum_{r=1}^N \kappa(\lambda^{(r)})} \cdot \prod_{r<s}^{N} q^{\frac{N}{4}\left(\kappa(\lambda^{(r)})-\kappa(\lambda^{(s)})\right)} = q^{-\frac{N}{2}\sum_{r=1}^N r\kappa(\lambda^{(r)})}, \qquad (5.18)
$$

while the exponents of Q_{F_r} become

$$\prod_{r<s}^{N}\left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{|\lambda^{(s)}|} = \prod_{r=1}^{N-1} Q_{F_r}^{r\sum_{s=r+1}^{N}|\lambda^{(s)}|}. \tag{5.19}$$

Together with the contributions of the Schur functions from (5.16) and (5.17) we finally obtain

$$\begin{aligned}
\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(s)}\},\{p_s\},Q) = & \pm Q^{\frac{N}{2}\sum_{r=1}^{N}|\lambda^{(r)}|} q^{-\frac{N}{2}\sum_{r=1}^{N} r\kappa(\lambda^{(r)})} \times \\
& \times \prod_{r=1}^{N-1} Q_{F_r}^{r\sum_{s=r+1}^{N}|\lambda^{(s)}|} \prod_{r=1}^{N} s_{\lambda^{(r)}}(q^{N\rho}) \times \\
& \times \prod_{r<s}^{N}\prod_{k=1}^{\infty}\left(1-\left(\prod_{t=r}^{s-1} Q_{F_t}\right) q^{Nk}\right)^{k} \times \\
& \times \sum_{\nu^{(1)},\cdots,\nu^{(N-1)}}\left(\prod_{t=1}^{N-1} Q_{F_t}^{|\nu^{(t)}|}\right) s_{\nu^{(1)}}(q^{N\left(\lambda^{(1)}+\rho\right)}) s_{\nu^{(N-1)}}(q^{N(\widetilde{\lambda^{(N)}}+\rho)}) \times \\
& \times \sum_{\chi^{(1)},\cdots,\chi^{(N-2)}} \prod_{t=1}^{N-2} s_{\nu^{(t)}/\chi^{(t)}}(q^{N(\widetilde{\lambda^{(t)}}+\rho)}) s_{\nu^{(t+1)}/\chi^{(t)}}(q^{N(\lambda^{(t)}+\rho)}).
\end{aligned} \tag{5.20}$$

Interpretation as five-dimensional SU(N) Yang-Mills. We fix the U(1)-charges p_r. The relevant field theory parameters are a_r, Λ and R, where a_r ($r = 1, \ldots, N$) are the VEVs of the adjoint scalar in the vector multiplet. We identify q, Q_{F_r} and Q with a_r, Λ and R as follows.

$$q = e^{-\frac{2}{N}R\hbar}, \qquad Q_{F_r} = e^{-2R(a_r-a_{r+1})}, \qquad Q = -(2R\Lambda)^2. \tag{5.21}$$

Since we have set $Q_{F_r} = q^{N(\widetilde{p}_r-\widetilde{p}_{r+1})}$ the above implies

$$a_r = \hbar\widetilde{p}_r. \tag{5.22}$$

By the identifications (5.21), $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}$ are translated to the instanton contributions in five-dimensional gauge theories. This follows by rephrasing the expression (5.15) in terms of the field theory parameters. In particular, we obtain

$$\sum_{\{\lambda^{(r)}\}} \mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\},\{p_r\},\,Q)^2 = Z_{5d\,\mathrm{SYM}}^{\mathrm{SU}(N)\,\mathrm{inst}}(\{a_r\};\,\Lambda,R,\hbar). \tag{5.23}$$

The r.h.s. is the instanton part of the partition function [2] for five-dimensional $\mathcal{N} = 1$ supersymmetric SU(N) Yang-Mills with the Chern-Simons term [13]. The Chern-Simons corrections come from the following factor in (5.15).

$$q^{-\frac{N^2}{4}\sum_{r=1}^{N}\kappa(\lambda^{(r)})} \prod_{1\le r<s\le N}\left(\prod_{t=r}^{s-1} Q_{F_t}\right)^{-\frac{|\lambda^{(r)}|-|\lambda^{(s)}|}{2}}, \tag{5.24}$$

which is also understood [17] as a part of the so-called framing factor of the topological string vertices [6]. The Chern-Simons coupling constant becomes $c_{cs} = N$.

Thanks to the identifications (5.21), the square of $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}$ takes a form analogous to (4.2). By using the expression (5.11) it becomes

$$\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\},Q)^2 = \prod_{r<s}^{N} \exp\Big\{ -g(p_r-p_s|\hbar,2R,\Lambda) - g(p_r-p_s|-\hbar,2R,\Lambda) \Big\}\,. \quad (5.25)$$

As is the case of SU(2), the functions $g(p_r-p_s|\hbar,2R,\Lambda)$ coincide with $\gamma_\hbar(\hbar(\tilde{p}_r-\tilde{p}_s)|2R,\Lambda)$ up to linear functions in $\hbar(\tilde{p}_r-\tilde{p}_s)$. Thus, by taking (5.22) into account, we obtain

$$\begin{aligned}\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}(\{p_r\},Q)^2 &= \exp\Big\{ -\Delta_N^{(0)}(\hbar,2R,\Lambda) - \sum_{r\neq s}^{N} \Delta_{N\,r,s}^{(1)}(\hbar,2R,\Lambda)(a_r-a_s) \Big\} \times \\ &\quad \times \exp\Big\{ -\sum_{r\neq s}^{N} \gamma_\hbar(a_r-a_s|2R,\Lambda) \Big\}\,. \end{aligned} \quad (5.26)$$

The expression (5.26) shows that the square of $\mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{pert}}$ is identified with the perturbative part of the partition function for five-dimensional supersymmetric SU(N) Yang-Mills. Let us recall that the perturbative part is introduced in (5.5) as the Boltzmann weight for the ground partition after removing the factor, $q^{-\frac{N^2}{12}\sum_{r=1}^{N}\tilde{p}_r^3}$. This removed factor turns to be the perturbative correction [18] from the Chern-Simons term. Together with (5.23) we finally find out

$$\begin{aligned} q^{-\frac{N^2}{6}\sum_{r=1}^{N}\tilde{p}_r^3}\mathcal{Z}_q^{SU(N)\,\mathrm{pert}}(\{p_r\},Q)^2 \sum_{\{\lambda^{(r)}\}} \mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\},\{p_r\},Q)^2 = \\ = Z_{5d\,\mathrm{SYM}}^{\mathrm{SU}(N)}(\{a_r\};\,\Lambda,R,\hbar)\,. \end{aligned} \quad (5.27)$$

The r.h.s. is the exact partition function [2] for the five-dimensional supersymmetric SU(N) Yang-Mills with the Chern-Simons term.

Similarly to the SU(2) case, the gauge theory is realized as the $\hbar \to 0$ limit of the above partition function. It is the thermodynamic limit with a_r, Λ and R fixed. The corresponding ground partition in (5.27) becomes very large and the U(1) instantons $\lambda^{(r)}$ provide the deformation of the ground partition. Hence we can consistently say that the ground partitions describe the perturbative regime of the Coloumb branch while their deformation is non-perturbative in the gauge theory.

Ground partitions and classical SU(N) geometries. The relevant non-compact Calabi-Yau threefold is an ALE space with A_{N-1} singularity fibred over $\mathbb{P}^1$. Type of the fibrations is labelled by the framing $m \in [0,\,N]$. We choose the framing to be $m = N$. The geometrical data are the Kähler parameters t_B and t_{F_r}, where $1 \le r \le N-1$. They correspond respectively to the Kähler volumes of the base $\mathbb{P}^1$ and the $N-1$ blow-up cycles of A_{N-1} singularity in the fibre.

The topological string amplitude on this local geometry is given in (A.1). We denote the amplitude as $Z_{\mathrm{string}}^{\mathrm{SU}(N)}(q,\{Q_{F_r}\},Q_B)$, where $q = e^{-g_{st}}$, $Q_{F_r} = e^{-t_{F_r}}$ and $Q_B = e^{-t_B}$.

By comparing (5.20) with (A.1), we see that the gauge instanton contribution (5.23) is converted to the string amplitude as follows.

$$\sum_{\{\lambda^{(r)}\}} \mathcal{Z}_q^{\mathrm{SU}(N)\,\mathrm{inst}}(\{\lambda^{(r)}\},\{p_r\},Q)^2 =$$
$$= \prod_{r<s}^{N}\prod_{k=1}^{\infty}\Big(1-\Big(\prod_{t=r}^{s-1}Q_{F_t}\Big)q^{Nk}\Big)^{2k} Z_{\mathrm{string}}^{\mathrm{SU}(N)}(q^N,\{Q_{F_r}\},Q^N)\,, \tag{5.28}$$

where we put $Q_{F_r} = q^{N(\widetilde{p}_r-\widetilde{p}_{r+1})}$. This is in accord with the result [7, 8] that the topological string amplitude on this local Calabi-Yau geometry leads to the instanton part of the partition function of five-dimensional supersymmetric SU(N) Yang-Mills.

Due to the identification $\hbar \sim g_{st}$, the thermodynamic limit is in the perturbative regime of string theory. The ground partitions at the thermodynamic limit should be interpreted as classical objects in string theory. The relations $t_{F_r} = 2R(a_r - a_{r+1})$, which follow from (5.28), suggest that the SU(N) ground partitions describe the resolutions of A_{N-1} singularity in this local geometry. This identification allows us to interpret the perturbative Chern-Simons correction as the triple intersections of the four-cycles [18] in the local geometry.

A. Topological string amplitudes

We present the topological string amplitudes related to four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories. The geometric engineering [5] dictates that four-dimensional $\mathcal{N}=2$ supersymmetric SU(N) Yang-Mills is realized by an ALE space with A_{N-1} singularity fibered over $\mathbb{P}^1$. The fibration over $\mathbb{P}^1$ is labeled by an integer $m \in [0, N]$, which is often called the framing. We consider the case of $m = N$. The geometric data turn to be related with the field theory parameters. The Kähler parameter t_B of the base $\mathbb{P}^1$ is proportional to $1/g^2$, where g is the gauge coupling constant at the string scale. The gauge instanton effect is weighted with $e^{-c_2/g^2} \sim \Lambda^{2Nc_2}$. This leads to $e^{-t_B} \sim \Lambda^{2N}$. The Kähler parameters t_{F_r} of the blow-up cycles in the fibre are proportional to a_r, the VEVs of the adjoint scalar in the vector multiplet.

The topological vertex [6] provides a powerful method to compute all genus topological A-model partition functions on local toric Calabi-Yau threefolds. The computations are carried out by using diagrammatic techniques like the Feynman rules. The diagrams are the dual toric diagrams consisting of trivalent vertices. Topological string amplitudes on $\mathbb{C}^3$ with open-string boundary conditions are attached to the vertices. They are derived based on the large N duality [19] and the boundary conditions turn out to be fixed by three partitions. Gluing the vertices by a certain rule, one obtains the topological string amplitudes on the threefolds.

The diagram which describes the above local geometry for the SU(N) gauge theory is depicted in figure 9. The method of the topological vertex gives the amplitude in the

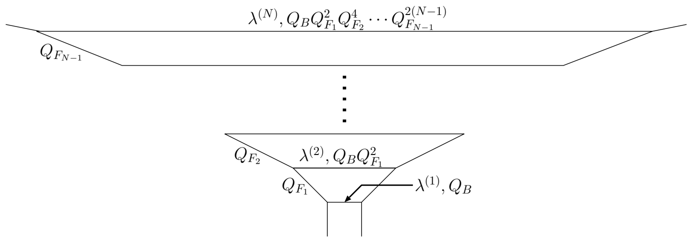


Figure 9: The diagram for the local geometry of the SU(N) gauge theory. $Q_i = e^{-t_i}$, where $t_i = t_B, t_{F_r}$ $(1 \le r \le N-1)$. Partitions $\lambda^{(r)}$ are attached to the corresponding $\mathbb{P}^1$s.

following form.[3]

$$
\begin{aligned}
&Z^{\mathrm{SU}(N)}_{\text{string}}(q, \{Q_{F_r}\}, Q_B) = \\
&= \sum_{\lambda^{(1)},\cdots,\lambda^{(N)}} \left[\begin{array}{l}
Q_B^{\frac{1}{2}\sum_{r=1}^{N}|\lambda^{(r)}|} q^{-\frac{1}{2}\sum_{r=1}^{N} r\kappa(\lambda^{(r)})} \displaystyle\prod_{r=1}^{N-1} Q_{F_r}^{r\sum_{s=r+1}^{N}|\lambda^{(s)}|} \prod_{r=1}^{N} s_{\lambda^{(r)}}(q^{\rho}) \\
\times \displaystyle\sum_{\nu^{(1)},\cdots,\nu^{(N-1)}} \left(\prod_{t=1}^{N-1} Q_{F_t}^{|\nu^{(t)}|} \right) s_{\nu^{(1)}}(q^{\lambda^{(1)}+\rho}) s_{\nu^{(N-1)}}(q^{\widetilde{\lambda^{(N)}}+\rho}) \\
\times \displaystyle\sum_{\chi^{(1)},\cdots,\chi^{(N-2)}} \prod_{t=1}^{N-2} s_{\nu^{(t)}/\chi^{(t)}}(q^{\widetilde{\lambda^{(t)}}+\rho}) s_{\nu^{(t+1)}/\chi^{(t)}}(q^{\lambda^{(t)}+\rho})
\end{array} \right]^2 ,
\end{aligned}
\tag{A.1}
$$

where the parameters q, Q_{F_r} and Q_B are given by

$$
q = e^{-g_{st}}, \qquad Q_{F_r} = e^{-t_{F_r}}, \qquad Q_B = e^{-t_B}. \tag{A.2}
$$

(A.1) for the case of SU(2) becomes as follows; $(Q_F = Q_{F_1})$

$$
Z^{\mathrm{SU}(2)}_{\text{string}}(q, Q_F, Q_B) = \sum_{\lambda^{(1)},\lambda^{(2)}} \left[\begin{array}{l}
Q_B^{\frac{1}{2}(|\lambda^{(1)}|+|\lambda^{(1)}|)} q^{-\frac{1}{2}(\kappa(\lambda^{(1)})+2\kappa(\lambda^{(2)}))} Q_F^{|\lambda^{(2)}|} \times \\
\times s_{\lambda^{(1)}}(q^{\rho}) s_{\lambda^{(2)}}(q^{\rho}) \displaystyle\sum_{\nu} Q_F^{|\nu|} s_{\nu}(q^{\lambda^{(1)}+\rho}) s_{\nu}(q^{\widetilde{\lambda^{(2)}}+\rho})
\end{array} \right]^2 . \tag{A.3}
$$

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